Groups meet Analysis: the Fourier Algebra

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Leeds

York, June 2013

Colloquium talk

- So I believe this is a talk to a general audience of Mathematicians.
- Some old advice for giving talks: the first 10 minutes should be aimed at the janitor; then at undergrads; then at graduates; then at researchers; then at specialists; and finish by talking to yourself.
- The janitor won't understand me; and I'll try not to talk to myself.
- I'm going to try just to give a survey talk about a particular area at the interface between algebra and analysis.
- Please ask questions!

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Fourier transform

Let f be a "well-behaved" function on the real line. Then the Fourier transform of f is

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi ixt}\, dt.$$

(You have to put a 2π somewhere!)

Then we can reconstruct f from $\hat{f}$ by

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(x)\, e^{2\pi ixt}\, dx.$$

- A basic tool in "applied" mathematics which we teach to undergraduates.
- Appears in probability theory as the Characteristic Function.

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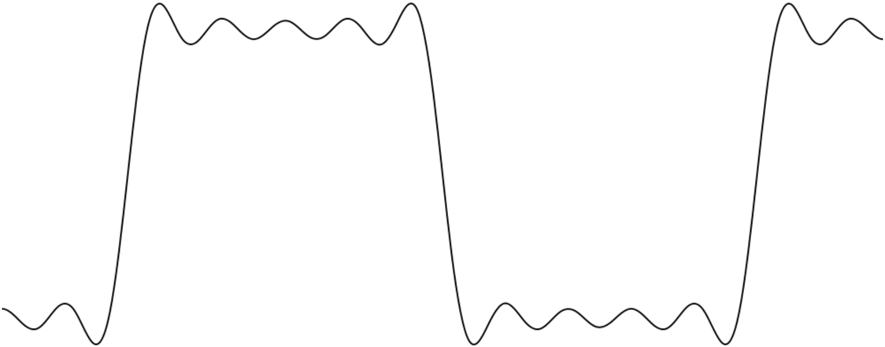

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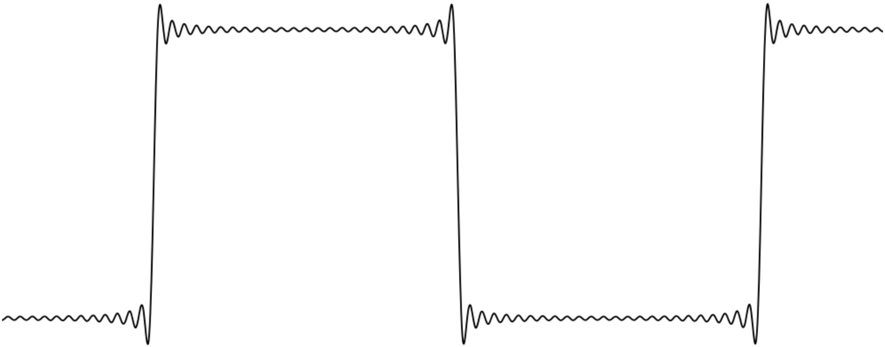

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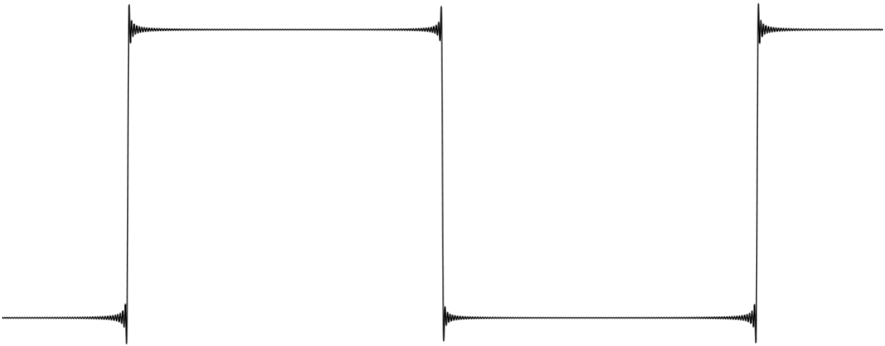

Fourier series

Given a periodic function $f : \mathbb{R} \to \mathbb{C}$ the Fourier series of f is $(\hat{f}(n))_{n\in\mathbb{Z}}$ where

$$\hat{f}(n) = \int_0^1 f(\theta) e^{-2\pi i n\theta} \, d\theta.$$

We have the well-known “reconstruction”:

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta}.$$

Of course, a great deal of classical analysis is concerned with the question of in what sense does this sum actually converge?

Convergence

$$f(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n\theta} \;??$$

- If f is twice continuously differentiable, then the sum converges uniformly to f (that is, $\lim_{N\to\infty} \sum_{n=-N}^{N}$).
- (Fejer) If f is continuous, and we take Cesaro means, then we always get (uniform) convergence.
- (Kolmogorov) There is a (Lebesgue integrable) function f such that the sum diverges everywhere.
- (Carleson) If f is continuous then the sum converges almost everywhere.

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A more "global" perspective

Don't want to look at single functions in isolation; but rather at spaces of functions.

Let's consider $L^2([0,1])$; that is, functions f with $\int_0^1 |f|^2 < \infty$.

- This is a vector space.
- $\|f\| = \left(\int_0^1 |f|^2 \right)^{1/2}$ is a norm.
- So we get a metric $d(f,g) = \|f - g\|$.
- With some help from Lebesgue, we get a complete space (so a Banach space; even a Hilbert space).

(Parseval) In the Banach space $L^2([0,1])$, we always have that

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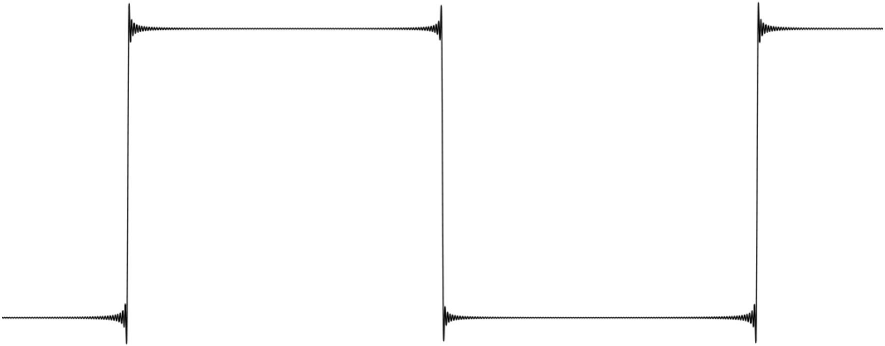

Where does it all come from?

Why do we link periodic functions with the integers, using $e^{2\pi i \cdot}$?

- Consider $[0, 1)$ with addition modulo 1.
- Same as $\mathbb{R}/\mathbb{Z}$; hence why we get *periodic* functions.
- This is the same as the "circle group" $\mathbb{T}$ (where we identify $t \in [0, 1)$ with the point on the circle at angle $2\pi t$).
- Let's consider *continuous* group homomorphisms $\phi : [0, 1) \to \mathbb{T}$. So $\phi(s + t) = \phi(s)\phi(t)$.
- These must be of the form

  $$\phi(t) = e^{2\pi itn},$$

  for some $n \in \mathbb{Z}$.
- The "Pontryagin dual" of $[0, 1), +$ is $\mathbb{Z}$.

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Abelian groups

In fact, we can always do this for a (locally compact) *abelian* group. Write $\hat{G}$ for the dual of G.

- The dual to $\mathbb{Z}$ is $[0, 1) \cong \mathbb{T}$ again.
- In general, always true that the dual of the dual is what you started with (biduality theory).
- Any continuous homomorphism $\phi : \mathbb{R} \to \mathbb{T}$ is of the form $\phi(t) = \exp(2\pi itx)$ for some $x \in \mathbb{R}$.
- So the dual of $\mathbb{R}$ is $\mathbb{R}$ (scaled by 2π).
- For any abelian group we have a Fourier transform which has all the properties we expect:
  - Plancheral– $L^2(\hat{G})$ and $L^2(G)$ are isometric.
  - Algebra property: The Fourier transform converts convolution of functions on G into pointwise multiplication of functions on $\hat{G}$.

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Convolutions

Remember that the *convolution* of functions f, g on $\mathbb{R}$ is

$$f * g(s) = \int_{-\infty}^{\infty} f(t)g(-t+s)\,dt.$$

Then if $h = f * g$ then $\hat{h} = \hat{f}\hat{g}$.

- The integral won't always converge.
- Let's restrict to $L^1(\mathbb{R})$ (f such that $\|f\|_1 = \int |f| < \infty$).
- Then $L^1(\mathbb{R})$ with convolution becomes an algebra: we even get $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.
- Let $A(\mathbb{R}) = \{\hat{f} : f \in L^1(\mathbb{R})\}$.
- Then $A(\mathbb{R})$ is an algebra for the pointwise product– indeed, it's just $L^1(\mathbb{R})$ viewed in a different way.
- (Riemann-Lebesgue) $A(\mathbb{R})$ consists of continuous functions which decay to 0 at ∞.
- But not *all* such functions. However, we get "enough".

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- The integral won't always converge.
- Let's restrict to $L^1(\mathbb{R})$ (f such that $\|f\|_1 = \int |f| < \infty$).
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- But not *all* such functions. However, we get "enough".

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Remember that the *convolution* of functions f, g on $\mathbb{R}$ is

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General abelian groups G

To be formal, $\hat{G}$ is the collection of continuous homomorphisms $\phi : G \to \mathbb{T}$ (characters).

- The product on $\hat{G}$ is pointwise:

$$(\phi\psi) : G \to \mathbb{T}; \quad s \mapsto \phi(s)\psi(s).$$

- Give $\hat{G}$ the topology of compact convergence.
- Then $\hat{G}$ is locally compact, so has a Haar measure.

We get Fourier Transforms: for f a function on G and g a function on $\hat{G}$, define

$$\mathcal{F}f(\phi) = \int_G f(s)\overline{\phi(s)}\, ds, \qquad \mathcal{F}^{-1}g(\phi) = \int_{\hat{G}} g(\phi)\phi(s)\, d\phi.$$

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- For any G the Fourier transform gives an isomorphism $L^1(\hat{G}) \to A(G)$.
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- So $A(\mathbb{T})$ is the Fourier transform of $\ell^1(\mathbb{Z})$– those periodic functions with “absolutely convergent Fourier series”.
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- We can write any $f \in L^1(\hat{G})$ as the pointwise product of two $L^2(G)$ functions, e.g.
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Some examples

Question:

How much of G does $A(G)$ "remember"?

- Suppose that G is finite.
- $A(G)$ consists of enough functions to separate the points of G.
- As G is finite, we simply get *all* functions $G \to \mathbb{C}$.
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Need to throw in some analysis

What's missing is that $A(G)$ also carries a norm:

$$\|a\|_{A(G)} = \inf\left\{\|f\|_2\|g\|_2 : a = f * g\right\}.$$

- Then we get $\|ab\|_{A(G)} \leq \|a\|_{A(G)}\|b\|_{A(G)}$.
- Also $A(G)$ is complete (a Banach algebra).

Theorem (Walter, 1972)

If $A(G)$ and $A(H)$ are isometrically isomorphic then G is isomorphic to either H or the opposite to H (same group, with product reversed).

- Indeed, you can actually write down what the isomorphism must look like.
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If time allows, some representation theory

- A representation of a group is a continuous homomorphism from G to $U(n)$, the unitary group.
- As $U(1) \cong \mathbb{T}$, the one-dimensional representations of G are just the characters. For an abelian group can always diagonalise.
- If G is infinite, then often we need to look at infinite-dimensional representations.
- This is a continuous homomorphism ϕ from G to $U(H)$, the unitary group of a Hilbert space. Continuous means

$$s_n \to s \text{ in } G \implies \phi(s_n)\xi \to \phi(s)\xi \text{ for all } \xi \in H.$$

- Important example: the left-regular representation λ of G on $L^2(G)$.

$$\lambda(s)f : t \mapsto f(s^{-1}t) \qquad (f \in L^2(G), s, t \in G).$$

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Coefficients

Given a representation $\phi : G \to U(H)$, a *coefficient* of ϕ is a (continuous) function on G of the form

$$f(s) = \langle \phi(s)\xi, \eta \rangle.$$

Let's consider a coefficient of the left-regular representation

$$\langle \lambda(s)f, g \rangle = \int_G f(s^{-1}t)\overline{g(t)}\, dt = \int_G \overline{g(t)}\check{f}(t^{-1}s)\, dt = \overline{g} * \check{f}(s).$$

Here $\check{f}(r) = f(r^{-1})$.
(I lied before: for many groups G it's not true that $f \in L^2(G) \implies \check{f} \in L^2(G)$.)
So $A(G)$ equals the collection of coefficients of λ.

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(I lied before: for many groups G it's not true that $f \in L^2(G) \implies \check{f} \in L^2(G)$.)

So $A(G)$ equals the collection of coefficients of λ.

Coefficients

Given a representation $\phi : G \to U(H)$, a *coefficient* of ϕ is a (continuous) function on G of the form

$$f(s) = \langle \phi(s)\xi, \eta \rangle.$$

Let's consider a coefficient of the left-regular representation

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Why an algebra?

Fix a representation ϕ. If we take the linear span of coefficients of ϕ we get a vector space of functions on G, say A_ϕ.

If we (pointwise) multiply $a \in A_\phi$ and $b \in A_\psi$ then

$$a(s)b(s) = \langle \phi(s)\xi_1, \eta_1 \rangle \langle \psi(s)\xi_2, \eta_2 \rangle = \langle (\phi(s) \otimes \psi(s))(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle.$$

So multiplication corresponds to tensoring representations.

Theorem (Fell's absorption principle)

For any ϕ, we have that $\lambda \otimes \phi$ is isomorphic to $\lambda \otimes 1_H$, that is, a direct sum of copies of λ.

Corollary

For $a \in A(G) = A_\lambda$ and $b \in A_\phi$, we have that $ab \in A_\lambda$. In particular, $A(G)$ is an algebra.

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Finally: why no linear span?

I defined $A(G)$ as functions of the form $f * g$; no linear span!

- This is actually true, but to prove requires a lot of modern functional analysis.
- Take the linear span of $\{\lambda(s) : s \in G\}$, which gives us an algebra of operators on the Hilbert space $L^2(G)$.
- Closing this with respect to a suitable topology gives the *group von Neumann algebra* $VN(G)$.
- [Dixmier; Tomita, Takesaki] shows that $A(G)$ can be identified with the "predual" of $VN(G)$ and that hence every member of $A(G)$ is of the simple form claimed.

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Example: compact groups

Let G be compact.

- The representation theory is particularly nice.
- The *irreducible* representations are finite-dimensional; let $\hat{G}$ be the classes of irreducibles.
- Every representation is (isomorphic to) the direct sum of irreducibles.

In particular, λ decomposes as

$$\lambda = \bigoplus_{\phi \in \hat{G}} d_\phi \phi.$$

Then, as a Banach space

$$A(G) = \ell^1 - \bigoplus_{\phi \in \hat{G}} d_\phi (\mathbb{M}_{d_\phi}, \text{trace-norm}).$$

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My actual research

I'm interested in "non-commutative geometry/topology".

- Idea is that "spaces" correspond to "commutative algebras" (e.g. Gelfand theory of C^*-algebras).
- So then non-commutative algebras correspond to "non-commutative spaces".
- For abelian groups, we have Pontryagin duality.
- If I look at algebras, then $L^1(\hat{G}) = A(G)$.
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- This still makes sense if G is not abelian.
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